



Thermal Stresses in Thin-Walled Cylinders

By CLARENCE H. KENT,¹ FAYETTEVILLE, ARK.

The author, using Föppl's method, investigates the stresses in (1) a cast-iron cylinder 30 in. long, 20 in. mean diameter, and 1 in. in wall thickness, with the temperature 100 deg. Fahr. higher inside than outside; (2) the same cylinder and temperature conditions, but with cylinder supported at the ends; and (3) a steel cylinder of same size with flat heads rigidly attached and subjected to 500 lb. per sq. in. internal pressure. Constant and variable temperature gradients are considered, both in axial and radial directions.



THE problem of the stresses in a long, hollow cylinder caused by a temperature gradient in the radial direction was first solved by T. M. C. Duhamel (1)² in 1838. The solution was presented in a memoir which laid the foundation for future study of thermal stresses. Later writers have brought the problem to its present-day form (5), (6), (10). The method is exact for an infinitely long cylinder of any thickness and any temperature distribution in the

radial direction only. It finds the principal planes of stress in such a cylinder to be in the radial, axial, and tangential directions with normal stresses acting on each, and shows that stresses on a plane perpendicular to the axis are independent of the position of this plane.

In order that the solution may be exact when a cylinder of finite length is chosen, boundary conditions require that external forces equal in value and distribution to the stresses in the axial direction act on the ends. In the usual case such forces will not be present, so that the stresses near the ends will be very different from those calculated by this method. The effect diminishes as we recede from the ends, and practically disappears at a point sufficiently far away. Some writers have assumed that this point was located at a distance from the ends equal to the thickness of the cylinder wall, but its location really depends on the ratio of this thickness to the cylinder diameter. For such examples as the liner of a Diesel-engine cylinder this effect does not disappear until we have moved from the ends a distance equal to about twelve times

the wall thickness. Since the tangential stresses at the ends may be as much as 35 per cent higher than those near the middle, it becomes important to investigate them.

For determining the value of these stresses in thin cylinders Dr. August Föppl (21) turned to a method which had been in use for some time in the case of cylinders subjected to internal or external pressure. This was to treat an element cut from the cylinder wall as a beam on an elastic foundation. The method is not exact because it disregards radial stress, but it is sufficiently accurate for cases in which the cylinder diameter is at least ten times the wall thickness.

Applying the method first to a cylindrical shell of finite length in which a temperature gradient exists in the direction of the radius only, it is found that the maximum tangential stress at the ends is 30 per cent higher than the similar stress in an infinite cylinder. In a cast-iron cylinder of 1 in. thickness, 20 in. mean diameter, and 30 in. length, the tangential stress at the ends produced by a temperature difference between inner and outer surfaces of 100 deg. Fahr. is 7850 lb. per sq. in. Away from the ends the value of both the tangential and axial stresses is 6000 lb. per sq. in., which is the same as that in a thin plate subjected to the same temperature gradient through its thickness and prevented from bending. This stress is independent of the thickness of the plate, which seems to contradict experience, as it is usually considered that a thicker plate will be subjected to higher thermal stresses. For the same rate of heat flow this is true, but in that case the temperature difference will be higher across a thick plate than across a thin one, and thus give rise to higher stress. The cylinder walls are deflected outward 0.0022 in. at the ends, which is the cause of the increase in tangential stress at this point. Fig. 6 shows the deflection of the walls and the distribution of tangential and axial stresses in the exterior surface.

The actual curve assumed by the cylinder walls has the form of a damped wave which is identical with that of a beam on an elastic foundation acted upon by a moment at the ends. In the case of the cylinder a small longitudinal strip of the walls represents the beam, and its elastic support is furnished by the resistance of the wall as a whole to expansion or contraction. Fig. 6 shows clearly how the moment produced by the thermal stresses has bent the end of the wall outward, and how this effect is damped out as one moves away from the ends.

If the deflection at the ends of such a shell is prevented by rigid supports such as those shown in Fig. 8, tangential stresses will be reduced. In the case of the shell just considered this reduction amounts to 43 per cent, as the curves in Fig. 9 indicate. It is noticed that the axial and tangential stresses reach a constant value of 6000 lb. per sq. in. at a point about 12 in. from the ends in both Figs. 6 and 9.

The supports at the ends do not entirely eliminate deflection, but they reduce its maximum value to less than 0.001 in. As the maximum axial stress is only 7 per cent above the normal "thin plate" value, one may conclude that for all practical purposes a cylinder so supported may be treated as of infinite length. The stresses in the wall will then be the same as those in the thin plate. Although the method does not apply to a thick cylinder, we may reason by analogy that the usual stress formulas for an infinite cylinder may be used with an equal degree of accuracy for a finite cylinder provided with similar end supports.

¹ Head of Department of Mechanical Engineering, University of Arkansas. Mem. A.S.M.E. Professor Kent was graduated from Purdue University in 1915. His experience comprises three years with the Erie R. R.; two years as 1st Lieut. of Engineers, U. S. Army; three years with the Westinghouse Elec. & Mfg. Co.; eight years as associate professor of Mechanical Engineering, University of Nevada; and, since 1928, as professor and head of department of mechanical engineering at the University of Arkansas. He has received the degrees of M.E. from Purdue (1927) and of M.S. from Iowa State College (1928).

² Numbers in parentheses indicate references in the bibliography following Appendix No. 2.

For presentation at the National Applied Mechanics Meeting, Purdue University, Lafayette, Ind., June 15 and 16, 1931, of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS. [Selected from a dissertation prepared under the direction of Prof. S. Timoshenko for presentation to the Faculty of the University of Michigan in partial fulfillment of the requirements for the degree of Doctor of Philosophy.]

NOTE: Statements and opinions advanced in papers are to be understood as individual expressions of their authors, and not those of the Society.

The stress will be reduced, of course, by any support whether or not it is rigid, and its effect can be computed if its elastic properties are known. Obviously the stress at the ends under such conditions will fall somewhere between the 7850 lb. per sq. in. value of Fig. 6 and that of 4500 lb. per sq. in. of Fig. 9, with the values for the middle portion of the cylinder remaining at 6000 lb.

In Example 3 the method is applied to a steel cylinder of the same dimensions and having the same temperature difference as the cast-iron cylinders considered. We now assume that heads of the same material as the walls are rigidly attached; that is, there is no bending in the joint between the head and the walls. An internal fluid pressure of 500 lb. per sq. in. is now introduced, though to avoid unnecessary complications in the calculations no opening is assumed. Such an opening should be placed somewhere within the middle 10 in. of the wall, where the cylindrical form is not interfered with by the application of heat or pressure.

The head applies both a moment and a shear to the wall whose value is calculated from the slope of the head at the joint and its expansion in a radial direction under load. The head tends to be deflected outward by the pressure, but inward by the temperature strain and action of the walls. The tensile stress along any radius in the exterior of the head reaches a maximum of 28,000 lb. per sq. in. at the center for a flat head $1\frac{1}{2}$ in. thick.

The curves of Fig. 11 indicate a complex stress distribution in the walls. The internal axial stress which normally is compression is changed to tension at the ends by the moment of the cylinder heads and reaches a maximum value of 26,550 lb. per sq. in. The tangential stresses at the ends are kept low by the action of the heads in limiting deflection.

At the middle of the cylinder the tangential tension reaches a maximum of 20,300 lb. per sq. in. on the exterior, 5000 lb. of which is produced by the internal pressure and the rest is due to the temperature difference. Axial stresses here are lower since the internal pressure has a smaller effect on them. The deflection curve is a damped wave shifted from its normal position by the action of the moment and shear at the origin. At the middle the effect of the internal pressure is to prevent the deflection returning to zero as it did in previous cases.

This example is intended to illustrate the fact that a great variety of end conditions may be successfully handled by this method, as long as there is a means of arriving at an expression for the moment and shear exerted on the cylinder walls at the ends. A flanged connection at the ends would also afford a good example of this.

So far the discussion has been concerned with shells having a temperature gradient in the radial direction only. It happens in many applications that temperature gradients also exist in the axial direction. In an internal-combustion-engine cylinder, for example, heat is flowing radially through the walls, but the temperature of the interior varies at the same time in the axial direction.

A case of very localized heating, such as that which is produced in welding pipe, develops a large gradient in the axial direction. The method used in this paper is quite easily extended to cases where the temperature along the axis varies with integral powers of the axial coordinate. This gives sufficient latitude for handling most practical cases.

Suppose in the cast-iron cylinder of the first example, the interior temperature varies linearly along the axis from zero to 100 deg. Fahr., while the exterior is held at zero. Since the cylinder walls are thin, a linear distribution can still be assumed along the radius. Fig. 15 shows the deflection and the stress distribution for this case. The maximum stress at the warmer

end is slightly less than that of Fig. 6, and the maximum axial stress is some 1400 lb. per sq. in. less.

Any constant temperature difference may be superposed directly on the curves of Fig. 15 to find the combined effect. If, for example, the internal temperature varies linearly over the length from 100 deg. Fahr. to 200 deg. Fahr. while the exterior is kept at zero, the stress and deflection curves are obtained by adding the curves of Fig. 6 to those of Fig. 15.

If the temperature varies with the second power of the axial coordinate instead of the first, the problem may be handled in the same way. The stress curves obtained would differ but little from those of Fig. 15 at the end sections. The middle portion would be represented by a parabola instead of a straight line.

A case of this type of temperature distribution occurs in the cylinder liner of a Diesel engine. From experimental measurements of temperatures obtained by Dr. Eichelberg (11) and Dr. Letson (12) the temperature curves of Fig. 16 were drawn. The measurements were made in each case by thermocouples embedded in the liner at a distance of 5 mm. from each surface. The equations of the curves are of the second degree in x , and the stresses produced may be calculated by this method. In Fig. 16 the liner is shown pressed into its casing, which is assumed to exert a sufficient moment and shear on the liner to bring the deflection and slope at this end to zero.

The maximum stress of approximately 6500 lb. per sq. in. occurs at the end. As we have seen before, when the deflection of the end is prevented the cylinder behaves much as if it were infinite in length. The stresses obtained by the usual formulas applying to infinite cylinders are about 150 lb. per sq. in. below the value just given. The actual stress will depend upon the exact amount of the deflection permitted at the end. The assumptions, first of free ends and then of fixed ends, establish the limits between which any particular case might fall. Normally the end conditions will be much nearer to the fixed case.

In the example chosen the sharp peak of stress at the origin could be much reduced by designing the press fit of the liner in its casing to permit an outward slope of about 0.001 in. per inch at the origin and to keep the deflection of the wall beyond its normal position as low as possible. To see the effect of this, imagine that the maximum tangential stress in Fig. 15 is reduced to about that at the origin in Fig. 9. If the moment is removed the axial stresses will be reduced to zero at the end.

NOTATION

- a = linear expansion coefficient
- c = mean radius of cylindrical shell
- E = modulus of elasticity
- e_t = unit tangential deformation
- h = thickness of cylindrical shell
- I = moment of inertia of strip of unit width and depth h
- K = modulus of foundation
- l = length of cylinder
- M = moment
- p = pressure, lb. per sq. in.
- r = radius of curvature
- σ_r = stress in radial direction
- σ_t = stress in circumferential direction
- σ_x = stress in axial direction
- T = temperature
- ΔT = temperature difference between interior and exterior surfaces of cylinder
- η = distance from neutral axis
- μ = Poisson's ratio
- $\phi; \psi; \theta; \alpha$ = functions of x , see Appendix No. 1.

HEAT FLOW

If the interior surface of a hollow cylinder be maintained at a temperature T_1 , and the exterior at a temperature T_2 , flow of heat takes place through the walls which after a sufficient time becomes a steady flow. The temperature of any point in the wall is then given by the solution of the differential equation:³

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} = 0$$

where T is the temperature of any point whose distance from the axis is r .

The solution is:

$$T = \frac{(T_2 - T_1) \log r + T_1 \log r_2 - T_2 \log r_1}{\log r_2 - \log r_1}$$

r_1 and r_2 being the inner and outer radii.

This discussion is confined to thin-walled cylinders in which the logarithmic distribution above becomes practically linear and is given by:

$$T = \frac{(T_2 - T_1)r + T_1 r_2 - T_2 r_1}{r_2 - r_1}$$

Let $r_2 - r_1 = h$, and $(r_1 + r_2)/2 = c =$ radius of mid-plane of cylinder wall. Let $\eta = r - c$; thus η is measured from the mid-plane and positive outward.

Let $T_2 - T_1 = \Delta T$, $T_1 = -\Delta T/2$, and $T_2 = +\Delta T/2$; thus the temperature is also measured from the mid-plane and positive outward. Since a uniform heating or cooling of the entire cylinder will have no effect on stresses, we may choose the temperature of the mid-plane as we like. For convenience let its temperature be zero. By these substitutions the temperature equation becomes:

$$T = \frac{\eta}{h} \Delta T$$

when the temperature of the external surface is higher than that of the internal. When this condition is reversed and the heat flow is away from the center the temperature of any point is given by:

$$T = -\frac{\eta}{h} \Delta T \dots \dots \dots [1]$$

This expression is the same as that for a flat plate of thickness h subjected to a constant temperature difference ΔT .

It is to be noted that neither the coefficient of linear expansion α nor the modulus of elasticity E remains constant when the temperature varies. For temperature differences not exceeding 200 deg. Fahr., however, this variation will not seriously affect the result. One may express α as a function of the temperature and thus take care of its variation for temperature differences exceeding that mentioned (9). It is questionable whether the

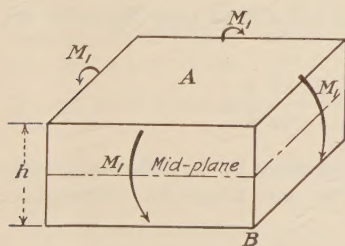


FIG. 1

³ Carslaw, p. 114.

CONSTANT TEMPERATURE GRADIENT IN RADIAL DIRECTION ONLY

Consider an element of a flat plate of thickness h and sides unity, shown in Fig. 1. Let a constant temperature difference ΔT be applied to the surfaces A and B so that the temperature of any point is a function of its distance from the mid-plane and is given by Equation [1]. The element will then be bent to a spherical surface of radius:⁴

$$\frac{1}{r} = \frac{\alpha \Delta T}{h} \dots \dots \dots [2]$$

This curvature will be removed by the application of the moments M_1 in Fig. 1, whose value is:

$$M_1 = -\frac{EI}{1 - \mu} \frac{\alpha \Delta T}{h} \dots \dots \dots [3]$$

The stresses produced by these moments are:

$$\sigma_x = \sigma_y = \frac{E \alpha \Delta T}{1 - \mu} \frac{\eta}{h} \dots \dots \dots [4]$$

If a similar element is cut from a thin cylindrical shell and heated in the same manner, Equation [2] represents the change in curvature which it will experience. The moments required to remove this change and the stresses resulting are given by Equations [3] and [4], except that in the cylinder σ_y is replaced by σ_t .

In order that we may have such a stress distribution in a cylinder of finite length we must assume that at the ends are applied uniformly distributed moments of value M_1 per inch of circumference, as shown in Fig. 2(a). In such a cylinder or in one of infinite length the stress distribution due to the application of the temperature difference ΔT is given by Equation [4].

In the usual case the ends of the cylinder are free. To accomplish this we shall have to apply to the cylinder ends a moment equal and opposite to M_1 as shown in Fig. 2(b). On the temperature stresses given by Equation [4] we must superpose the stresses produced by the application of the moment $-M_1$, whose value we shall now determine.

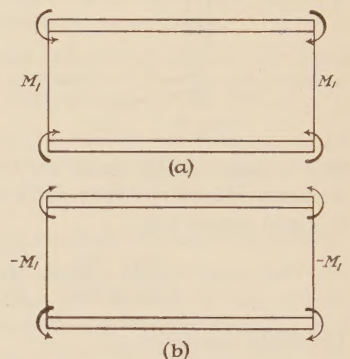


FIG. 2

Consider the cylinder at a uniform temperature which

we may take as zero. Let a strip of unit width of the thickness of the cylinder wall be cut from it as in Fig. 3. The moment $-M_1$ acts on the ends of this strip producing a deflection y , which is resisted by the radial component of tangential stresses set up by such movement, as shown in Fig. 4. Since all the strips composing the cylindrical shell are affected alike, there can be no shear between them.

The value of the radial component of these tangential stresses produced by a deflection y , is $(Eh/c^2)y$ per unit length of strip. The moment at any point x in the strip due to these forces can be computed from Fig. 5. If $K = Eh/c^2$, the force acting on the element of length du is $Kydu$ and its lever arm about the point x is $(x - u)$. The summation of moments about the point x gives:

$$M + M_1 + \int_0^x Ky(x - u)du = 0 \dots \dots \dots [5]$$

⁴ See Timoshenko, p. 487.

where M represents the internal moment at the cross-section mn , and is positive in a beam bent concave upward. The relation between moment and deflection in a thin strip is:

$$M = \frac{EI}{1 - \mu^2} \frac{d^2 y}{dx^2} \dots \dots \dots [6]$$

This modification of the usual bending formula is caused by the prevention of lateral contraction, since no change can take place in the angle θ of Fig. 3. Substituting the values of M and M_1 in [5]:

$$\frac{EI}{1 - \mu^2} \frac{d^2 y}{dx^2} - \frac{EI}{1 - \mu} \frac{a \Delta T}{h} + \int_0^x Ky(x - u) du = 0 \dots [7]$$

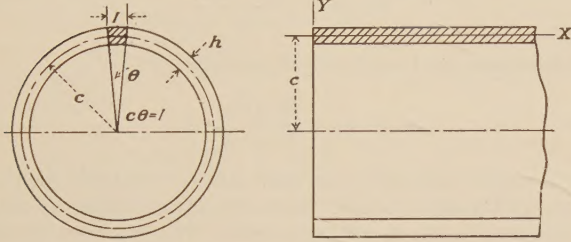


FIG. 3

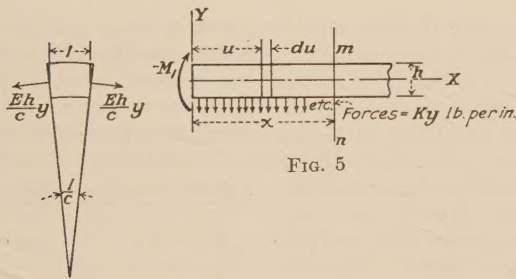


FIG. 4

This equation simply states the law of statics that the sum of moments acting about the point x must be zero. Differentiating this with respect to x :

$$\frac{EI}{1 - \mu^2} \frac{d^3 y}{dx^3} + \int_0^x Ky dx = 0 \dots \dots \dots [8]$$

Differentiating again:

$$\frac{EI}{1 - \mu^2} \frac{d^4 y}{dx^4} + Ky = 0 \dots \dots \dots [9]^5$$

These equations describe a beam on an elastic foundation bent by moments $\frac{EI}{1 - \mu} \frac{a \Delta T}{h}$ acting on the ends. The solution is:

$$y = e^{\beta x} (C_1 \cos \beta x + C_2 \sin \beta x) + e^{-\beta x} (C_3 \cos \beta x + C_4 \sin \beta x)$$

where $\beta = \sqrt[4]{\frac{K(1 - \mu^2)}{4EI}} = \sqrt[4]{\frac{3(1 - \mu^2)}{h^2 c^2}}$, and C_1, C_2 , etc., are

constants of integration.

We may dispose of two of these constants at once. As we move away from the ends of the cylinder the deflection of the wall, produced by a moment on the end, should decrease, and when x is large, should be practically zero. Under no circumstances could it show the rapid increase produced by the terms

multiplied by $e^{\beta x}$. Therefore we reason: $C_1 = C_2 = 0$. The deflection curve then takes the form:

$$y = e^{-\beta x} (C_3 \cos \beta x + C_4 \sin \beta x) \dots \dots \dots [10]$$

The analysis is valid for a cylinder of any length up to Equation [10]. In placing C_1 and C_2 equal to zero it was assumed that the cylinder was long enough to permit the deflection and slope of the wall to become practically zero at the middle. Specific examples will show what this length must be, but in general a distance equal to $2\pi/\beta$, which is the wave length of the harmonic form assumed by y will be sufficient.

To evaluate the constants C_3 and C_4 , we may compute from [10] the values of $d^2 y/dx^2$ and $d^3 y/dx^3$ for $x = 0$ and substitute them in Equations [7] and [8]. By differentiation we find:

$$\frac{d^2 y}{dx^2}(x = 0) = -2\beta^2 C_4$$

Placing $x = 0$ in [7] and substituting this value for $d^2 y/dx^2$,

$$C_4 = -\frac{1 + \mu}{2\beta^2} \frac{a \Delta T}{h}$$

By differentiation again, we obtain:

$$\frac{d^3 y}{dx^3}(x = 0) = 2\beta^3 (C_3 + C_4)$$

and from Equation [8]:

$$C_3 = \frac{1 + \mu}{2\beta^2} \frac{a \Delta T}{h}$$

Putting these values in [10],

$$y = \frac{1 + \mu}{2\beta^2} \frac{a \Delta T}{h} e^{-\beta x} (\cos \beta x - \sin \beta x) = \frac{1 + \mu}{2\beta^2} \frac{a \Delta T}{h} \psi \dots [11]$$

where $\psi = e^{-\beta x} (\cos \beta x - \sin \beta x)$

and:

$$\frac{d^2 y}{dx^2} = (1 + \mu) \frac{a \Delta T}{h} \phi \dots \dots \dots [12]$$

where $\phi = e^{-\beta x} (\cos \beta x + \sin \beta x)$.

As noted above, $l/2$ is assumed large enough to permit the effect of the moment on the ends to disappear at the middle of the cylinder. When both ends of the cylinder are free, the deflection, moment, stresses, etc., are symmetrical about the point $x = l/2$.

The axial stress in the shell owing to the deflection y is:

$$\sigma_x' = -\frac{M\eta}{I}$$

We may substitute in this the value of M from Equation [6], and of $d^2 y/dx^2$ from [12], giving

$$\sigma_x' = -\frac{Ea \Delta T}{1 - \mu} \frac{\eta}{h} \phi$$

To find the tangential stresses produced by the deflection we note that $e_t = y/c$, where e_t is the unit tangential deformation. From Hooke's law,

$$e_t = \frac{1}{E} (\sigma_t' - \mu \sigma_x')$$

⁵ An alternate derivation of this formula is given in Appendix No. 2.

Substituting the values of y and σ_x' :

$$\sigma_t' = \frac{Ea\Delta T}{h(1-\mu)} \left[\frac{1-\mu^2}{2\beta^2 c} \psi - \eta\mu\phi \right]$$

These stresses are due to the deflection y alone. We may now refer to Equation [4] and superpose the stresses existing in the shell owing to the temperature gradient to obtain the complete expressions for the stresses in a cylindrical shell, heated in the manner assumed and having ends free from external forces:

$$\sigma_x = \frac{Ea\Delta T}{1-\mu} \frac{\eta}{h} (1-\phi) \dots \dots \dots [13]$$

$$\sigma_t = \frac{Ea\Delta T}{1-\mu} \frac{\eta}{h} \left[1 - \mu\phi + \frac{1-\mu^2}{2\beta^2 c\eta} \psi \right] \dots \dots \dots [14]$$

Since $\phi = 1$ for $x = 0$, this expression for σ_x satisfies the condition of free ends. When $\eta = h/2$

$$\sigma_x = \frac{Ea\Delta T}{2(1-\mu)} (1-\phi)$$

$$\sigma_t = \frac{Ea\Delta T}{2(1-\mu)} \left[1 - \mu\phi + \frac{1-\mu^2}{\beta^2 hc} \psi \right]$$

$$\text{when } x = 0, \text{ max. } \sigma_t = \frac{Ea\Delta T}{2} \left(1 + \frac{1+\mu}{\sqrt{3(1-\mu^2)}} \right) \dots \dots \dots [15]$$

when x is large, $\phi = \psi = 0$, and

$$\sigma_x = \sigma_t = \frac{Ea\Delta T}{2(1-\mu)} \dots \dots \dots [4]$$

It may be shown in the usual way that the maximum value of σ_t occurs when $x = 0$ and $\eta = h/2$. The absolute value is less for $\eta = -h/2$.

If we assume $\mu = 0.25$, we may compare the value of σ_t at the ends with that for the region near the middle. Equation [4] then gives for the exterior surface, when x is large:

$$\sigma_t = \frac{Ea\Delta T}{2} (1.333)$$

For a similar point at the ends, Equation [15] gives:

$$\text{max. } \sigma_t = \frac{Ea\Delta T}{2} (1.744)$$

The peak stress at the ends is thus 30 per cent higher than the value near the middle.

The maximum value of σ_x occurs at $x = \pi/\beta$. Since the minimum value of $\phi = -0.043$, by referring to Fig. 17, the maximum axial stress will be only 4 per cent above the value given by Equation [4].

Example 1. As an illustration of the preceding section, assume a cast-iron cylinder of 20 in. mean diameter, 1 in. thickness, and 30 in. length. Let the inside temperature be 100 deg. Fahr. above the outside. Other constants are:

$E = 15 \times 10^6$ lb. per sq. in. $a = 6 \times 10^{-6}$ in. per in. per deg. Fahr.

$$\mu = 0.25 \quad \beta = \sqrt{\frac{3(1-\mu^2)}{h^2 c^2}} = 0.41 \text{ in.}^{-1}$$

The deflection at the ends is:

$$\text{max. } y = \frac{1+\mu}{2\beta^2} \frac{a\Delta T}{h} = 2.23 \times 10^{-3} \text{ in.}$$

The tangential and axial stresses away from the ends are given by Equation [4] and are: $\sigma_x = \sigma_t = \frac{Ea\Delta T}{2(1-\mu)} = 6000$ lb.

per sq. in. tension and compression on the outer and inner surfaces, respectively. A formula⁶ taking into account the true logarithmic temperature distribution in the cylinder walls and applying only to the portion distant from the ends gives, at the outer surface, $\sigma_t = \sigma_x = 5990$ lb. per sq. in.; and at the inner surface, $\sigma_t = \sigma_x = -6100$ lb. per sq. in. Thus for a ratio of $h/c = 0.1$, Equation [4] is very accurate.

The maximum tangential stress is:

$$\text{max. } \sigma_t = \frac{Ea\Delta T}{2} \left(1 + \frac{1+\mu}{\sqrt{3(1-\mu^2)}} \right) = 7850 \text{ lb. per sq. in.}$$

tension on the outer surface at the ends. The maximum axial stress occurs at $\pi/\beta = 7.67$ in. from each end, and its value is:

$$\text{max. } \sigma_x = 6270 \text{ lb. per sq. in.}$$

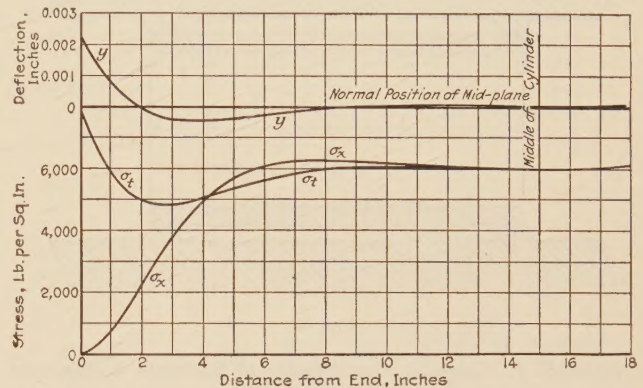


FIG. 6 TENSION IN OUTER FIBERS OF HOLLOW CAST-IRON CYLINDER (Length, 30 in.; diameter, 20 in.; thickness, 1 in. Temperature of internal surface, 100 deg. Fahr.; of external surface, 0 deg. Fahr.)

Fig. 6 shows the deflection of the cylinder wall from its normal position, and the variation in the tangential and axial stresses in the outer surface over one-half the length of the cylinder.

EFFECT OF CYLINDER HEADS OR OTHER END CONNECTIONS

An advantage of the method given is that it lends itself particularly well to the solution of the problem of the cylinder with end connections or cylinder heads exerting moments and loads on the ends and subjected at the same time to temperature gradients.

Let us again assume the cylindrical shell at a uniform temperature and with the moments and loads M_0 and q_0 per unit of circumference applied at the ends. A unit strip of this shell

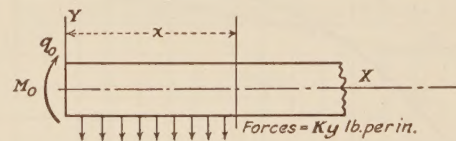


FIG. 7

will then be loaded as in Fig. 7. Equation [10] gives the deflection curve. Evaluating the constants for the end conditions in this case, we have:

$$C_3 = \frac{1-\mu^2}{2\beta^2 EI} \left(M_0 + \frac{q_0}{\beta} \right); \quad C_4 = -\frac{1-\mu^2}{EI} \frac{M_0}{2\beta^2}$$

⁶ Timoshenko, p. 554.

and:

$$y = \frac{1-\mu^2}{EI} \frac{1}{2\beta^2} \left(M_0 \psi + \frac{q_0}{\beta} \theta \right) \dots \dots \dots [16]$$

where, $\theta = e^{-\beta x} \cos \beta x$

From [16] we find:

$$\sigma_x = -\frac{12\eta}{h^3} \left(M_0 \phi + \frac{q_0}{\beta} \alpha \right) \dots \dots \dots [17]$$

where $\alpha = e^{-\beta x} \sin \beta x$, and

$$\sigma_t = \frac{E\eta}{C} + \mu \sigma_x \dots \dots \dots [18]$$

On these results will be superposed those in Equations [11], [13], and [14] to obtain the effects of a moment and load M_0 and q_0 combined with temperature difference.

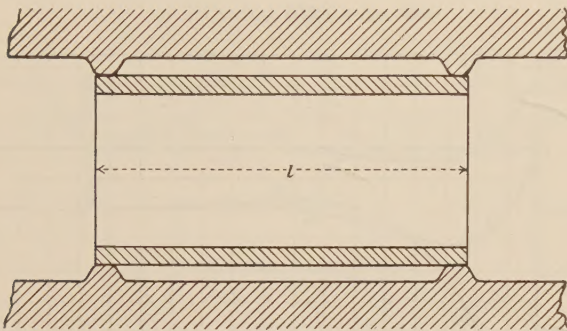


FIG. 8

Example 2. Consider the cylinder of Example 1 subjected to the same temperature difference but supported at the ends as in Fig. 8. Let these supports be sufficiently rigid that we may consider the deflection zero at the ends. Since there is no external moment at the ends, the load q_0 is determined from the fact that the sum of the deflections at the point $x = 0$ given by Equations [11] and [16] is zero:

$$0 = \frac{1+\mu}{2\beta^2} \frac{a\Delta T}{h} + \frac{1-\mu^2}{EI} \frac{q_0}{2\beta^2}$$

and

$$q_0 = -\frac{EI}{1-\mu} \beta \frac{a\Delta T}{h} = -410 \text{ lb. per in. of circumference}$$

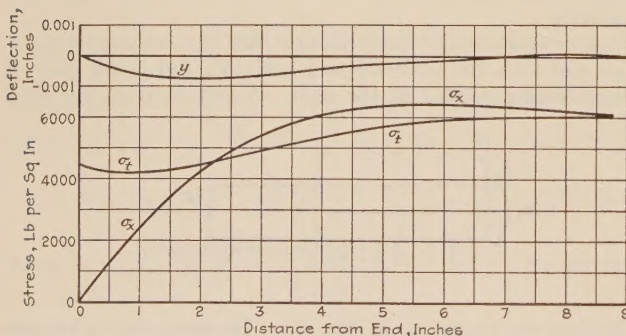


FIG. 9 TENSION IN OUTER FIBERS OF CYLINDER OF FIG. 6 SUPPORTED AS IN FIG. 8

To obtain the complete results for this case we may superpose on the curves of Fig. 6 the results of Equations [16], [17], and [18], in which $M_0 = 0$, and $q_0 = -410$ lb. per in. Fig. 9 shows the deflection and stresses produced in the cylinder walls.

Far from the ends the stresses are the same as those in Fig. 6. At the ends, however, the tangential stress has been reduced by the supports from a maximum of 7850 lb. per sq. in. in Fig. 6 to 4500 lb. per sq. in. in Fig. 9. This reduction depends only on the value of μ , and amounts to 42.7 per cent for $\mu = 0.25$.

Example 3. Assume a steel cylinder of the same dimensions as the cast iron one of Example 1 and heated in the same manner. In addition let it be subjected to an internal pressure of 500 lb. per sq. in. and fitted with flat heads rigidly attached to the walls. The heads are of the same material but h_1 in. thick. The temperature gradient is assumed to exist in the heads as well as the walls. The constants are:

$$\begin{aligned} E &= 30 \times 10^6 \text{ lb. per sq. in.} & h &= 1 \text{ in.} \\ a &= 6.7 \times 10^{-6} \text{ in. per in. per deg. Fahr.} & h_1 &= 1.5 \text{ in.} \\ \mu &= 0.3 \\ \beta &= 0.407 \text{ in.}^{-1} \end{aligned}$$

The deflection and stresses in the cylinder walls owing to the moment and shear at the ends produced by the heads are given in Equations [16], [17], and [18]. To these must be added the effects of the internal pressure, p lb. per sq. in., which produces a constant deflection pc^2/Eh . It increases the axial stress by an amount $pc/2h$, and the tangential stress by double this value. The deflection and stresses due to the temperature gradient may be taken from Equations [11], [13], and [14].

M_0 and q_0 are determined from a consideration of the cylinder head for which the axes are taken as in Fig. 10. The moment

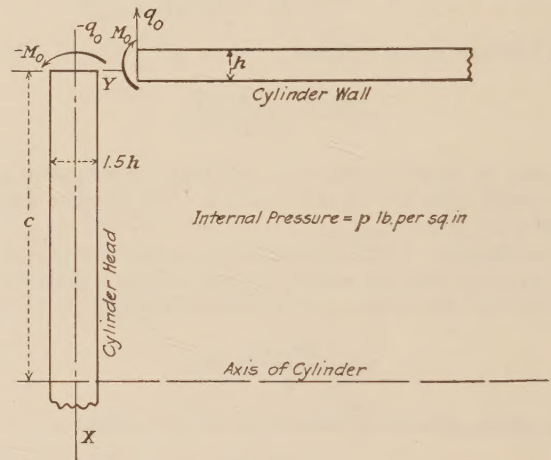


FIG. 10

and load at the edges are equal and opposite to those acting on the wall. The formulas applying to the head, which is treated as a thin plate, may be taken directly from textbooks covering this subject. The extension of the radius of the plate produced by the load $-q$ is $-\frac{q_0 c}{Eh_1} (1-\mu)$, which must be equal to the deflection of the walls at the end when the two are brought together. The wall deflection is given by Equations [11] and [16] plus pc^2/Eh . Equating these two expressions:

$$\frac{1+\mu}{2\beta^2} \frac{a\Delta T}{h} + \frac{pc^2}{Eh} + \frac{1-\mu^2}{EI} \frac{1}{2\beta^2} \left(M_0 + \frac{q_0}{\beta} \right) = -\frac{q_0 c}{Eh_1} (1-\mu) \dots \dots \dots [a]$$

The slope of the cylinder head will be influenced by three factors: the internal pressure; the temperature difference; and the moment $-M_0$ at the edges. The slope at the edges owing to the internal pressure is:

$$-\frac{3pc^3(1-\mu)}{2Eh_1^3}$$

That due to the moment is

$$\frac{12 M_0 c}{E h_1^3} (1 - \mu)$$

That due to the temperature gradient may be determined from the radius of curvature of a thin plate as given in Equation [2] and is: $\frac{a \Delta T}{h_1} c$. The slope of the wall at the ends is determined from the slope of Equations [11] and [16], when $x = 0$:

$$\frac{dy}{dx(x=0)} = -\frac{1-\mu^2}{EI} \frac{1}{2\beta^2} (2\beta M_0 + q_0) - \frac{1+\mu}{\beta} \frac{a \Delta T}{h}$$

When joined, the slope of the wall and the head will be the same:

$$-\frac{1-\mu^2}{EI} \frac{1}{2\beta^2} (2\beta M_0 + q_0) - \frac{1+\mu}{\beta} \frac{a \Delta T}{h} = -\frac{3pc^3(1-\mu)}{2Eh_1^3} + \frac{12M_0c}{Eh_1^3} (1-\mu) + \frac{a \Delta T}{h_1} c \dots [b]$$

Equations [a] and [b] determine M_0 and q_0 as follows:

$$M_0 = 180 \text{ lb-in. per inch of circumference.}$$

$$q_0 = -1590 \text{ lb. per inch of circumference.}$$

These values substituted in Equations [17] and [18] give the stresses in the wall resulting from the moment and shear produced by the head.

On these results we must superpose the deflection and stresses produced by the internal pressure as well as those resulting from the temperature gradient as given by Equations [11], [13], and [14]. The totals are:

$$y = \frac{1-\mu^2}{EI} \frac{1}{2\beta^2} (180\psi - 3880\theta) + \frac{1+\mu}{2\beta^2} \frac{a \Delta T}{h} \psi + \frac{pc^2}{Eh}$$

$$\sigma_x = \frac{Ea \Delta T}{1-\mu} \frac{\eta}{h} (1-\phi) - \frac{12\eta}{h^3} (180\theta + 3880\psi) + \frac{pc}{2h}$$

$$\sigma_t = Ea \Delta T \frac{\eta}{h} + E \frac{y}{c} + \mu \left(\sigma_x - \frac{pc}{2h} \right)$$

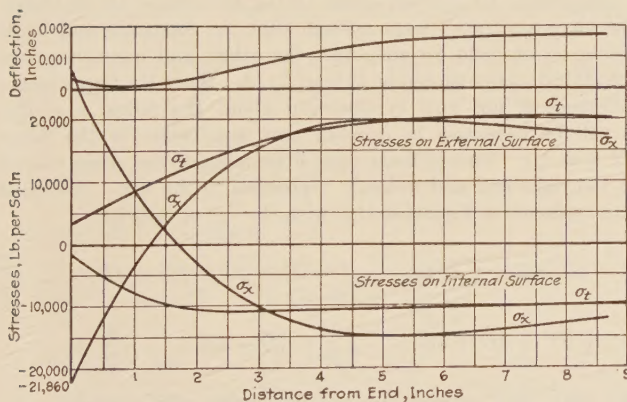


FIG. 11 CYLINDER WALL OF EXAMPLE 3

Fig. 11 shows the deflection of the cylinder wall and the stresses on the exterior and interior surfaces. For the plate, the effect of the internal pressure must be added to that of the moment and shear at the edges. The tensile stress along any radius of the outer surface is given by:

$$\sigma_r = \frac{3}{8} \frac{p}{h_1^3} (3 + \mu) (2cx - x^2) - \frac{6M_0}{h_1^2} - \frac{q_0}{h_1}$$

Fig. 12 shows the deflection and tensile stress along a radius of the outer surface of the cylinder head.

VARIABLE TEMPERATURE GRADIENT IN RADIAL DIRECTION

Assume now that a hollow cylinder of the form we have been considering is heated in such a manner that its temperature gradient in the radial direction varies along its length. As the cylinder walls are assumed thin, this gradient still remains linear. In order to prevent deflection of the mid-plane of the cylinder its temperature must remain at a constant value which

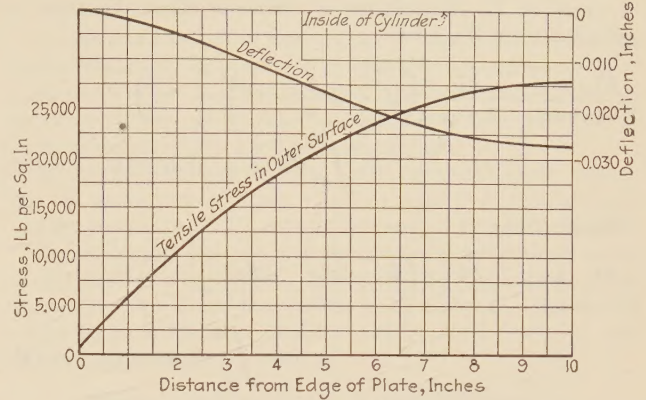


FIG. 12 CYLINDER HEAD, EXAMPLE 3

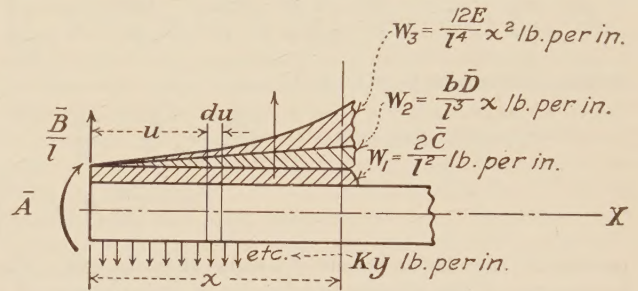


FIG. 13

we may take as zero. Let the temperature of the interior be given by the power series:

$$T_i = \frac{1}{2} \left[A + B \frac{x}{l} + C \frac{x^2}{l^2} + D \frac{x^3}{l^3} + E \frac{x^4}{l^4} + \dots \right] \dots [19]$$

where A, B , etc., are constants.

Then the temperature of the exterior at any value of x is equal to $-T_i$. The temperature difference between the two surfaces is $2T_i$. This difference produces the curvature of the element shown in Fig. 1, and the radius of curvature is found, as in Equation [2], to be:

$$\frac{1}{r} = \frac{2aT_i}{h}$$

Obviously this curvature is not now a constant in the X -direction. If the unit strip of Fig. 3 were bent by moments and loads to the form described by this equation, its internal moment at any point x would be:

$$M_x = \frac{EI}{1-\mu} \frac{1}{r} = \frac{EI}{1-\mu} \frac{a}{h} \left[A + B \frac{x}{l} + C \frac{x^2}{l^2} + \dots \right]$$

allowing no change in the angle θ , Fig. 3. To remove the curvature in the X -direction will require application of moments

and loads equal and opposite to those which produced the internal moment M_x . The moment at the ends is $M_{x(x=0)} = \bar{A}$, where $\bar{A}$ is written for $-\frac{EI}{1-\mu} \frac{a}{h} A$; the load at the end is $\frac{dM_x}{dx(x=0)}$

$= \frac{\bar{B}}{l}$; the distributed load is $\frac{d^2 M_x}{dx^2} = 2 \frac{\bar{C}}{l^2} + \frac{6\bar{D}}{l^3} x + \dots$ pounds per unit length. The sum of these separate moments and loads represents the total forces acting on the strip. Fig. 13 shows these forces, remembering that all the loads have a minus sign, and that the number of distributed loads depends on the number of terms in T_i . Summation of moments about the point x , as in Equation [7], gives:

$$\frac{EI}{1-\mu^2} \frac{d^2 y}{dx^2} - \frac{EI}{1-\mu} \frac{a}{h} \left(A + B \frac{x}{l} + C \frac{x^2}{l^2} + \dots \right) + \int_0^x K y (x-u) du = 0 \dots [20]$$

Differentiating,

$$\frac{EI}{1-\mu^2} \frac{d^3 y}{dx^3} - \frac{EI}{1-\mu} \frac{a}{h} \left(\frac{B}{l} + \frac{2Cx}{l^2} + \frac{3Dx^2}{l^3} + \dots \right) + \int_0^x K y dx = 0 \dots [21]$$

Differentiating again and transposing,

$$\frac{EI}{1-\mu^2} \frac{d^4 y}{dx^4} + K y = \frac{EI}{1-\mu} \frac{a}{h} \left[\frac{2C}{l^2} + \frac{6D}{l^3} x + \frac{12E}{l^4} x^2 + \dots \right]$$

Solving, after setting $C_1 = C_2 = 0$,

$$y = e^{-\beta x} (C_3 \cos \beta x + C_4 \sin \beta x) + \frac{ahc^2}{12(1-\mu)} \left[\frac{2C}{l^2} + \frac{6D}{l^3} x + \frac{12E}{l^4} x^2 + \dots \right]$$

provided T_i contains no powers of x higher than the fifth. The modification of this solution to take care of higher powers of x is easily made. Let us assume that T_i is limited to the first four terms of Equation [19] and obtain expressions for the stresses produced. Then:

$$\frac{d^2 y}{dx^2(x=0)} = -2\beta^2 C_4$$

From [20], when $x = 0$,

$$-\frac{EI}{1-\mu^2} 2\beta^2 C_4 - \frac{EI}{1-\mu} \frac{a}{h} A = 0$$

$$C_4 = -\frac{1+\mu}{2\beta^2} \frac{a}{h} A$$

$$\frac{d^3 y}{dx^3(x=0)} = 2\beta^3 (C_3 + C_4)$$

and from [21]:

$$C_3 = \frac{1+\mu}{2\beta^2} \frac{a}{h} \left(A + \frac{B}{\beta l} \right)$$

Then

$$y = \frac{1+\mu}{2\beta^2} \frac{a}{h} \left[\left(A + \frac{B}{\beta l} \right) \theta - A \alpha \right] + \frac{ahc^2}{12(1-\mu)} \left[\frac{2C}{l^2} + \frac{6D}{l^3} x \right] \dots [22]$$

$$\frac{d^2 y}{dx^2} = (1+\mu) \frac{a}{h} \left[\left(A + \frac{B}{\beta l} \right) \alpha + A \theta \right]$$

$$\sigma_x' = -\frac{Ea}{1-\mu} \frac{\eta}{h} \left[\left(A + \frac{B}{\beta l} \right) \alpha + A \theta \right] \dots [23]$$

$$\sigma_t' = \frac{Ey}{C} + \mu \sigma_x \dots [24]$$

To these stresses produced by the deflection y must be added those due to the temperature difference which are given by substituting $2T_i$ for ΔT in Equation [4]:

$$\frac{Ea}{1-\mu} \frac{\eta}{h} \left(A + B \frac{x}{l} + C \frac{x^2}{l^2} + D \frac{x^3}{l^3} \right) \dots [25]$$

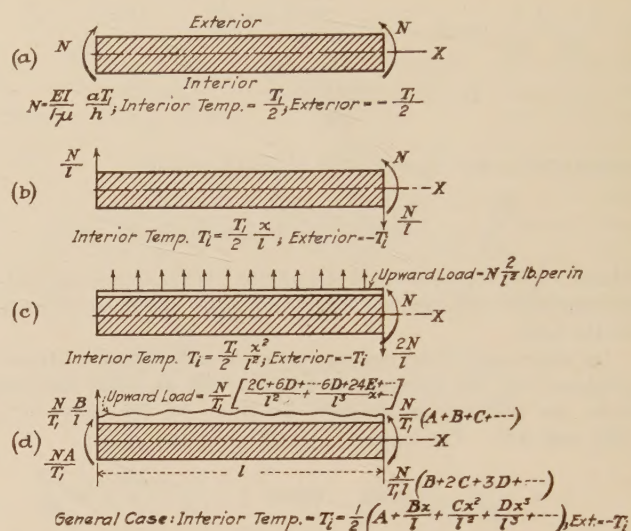


FIG. 14 MOMENTS AND LOADS ON SECTION OF CYLINDER WALL, 1 UNIT WIDE, PRODUCED BY THERMAL STRESSES. TEMPERATURE OF MID-PLANE = 0 DEG.

(In all cases the elastic foundation damps out deflection and stresses produced by moments and loads on ends before middle of cylinder is reached.)

Equations [19] to [24] apply to the left end of the cylinder. We have assumed l large enough that the moments and loads acting on one end do not affect the other, and have indicated that this distance should be at least $2\pi/\beta$. Since the temperature difference is not now symmetrical about the point $x = l/2$, it will be necessary to develop new equations for the range $l/2 < x < l$. This is easily done if we imagine the origin moved to the right end and measure x positive to the left. We must now replace x in Equation [19] with $(l-x)$ as follows:

$$T = [A + B + C + D + \dots] - [B + 2C + 3D + 4E + \dots] \frac{x}{l} + [C' + 3D + 6E + \dots] \frac{x^2}{l^2} - [D + 4E + \dots] \frac{x^3}{l^3} + \dots$$

We may then replace the brackets with new constants, A' , B' , etc., as:

$$T = [A'] - [B'] \frac{x}{l} + [C'] \frac{x^2}{l^2} - [D'] \frac{x^3}{l^3} + \dots$$

Equations [20] to [24] might then be rewritten by placing a prime on each of the constants, A , B , etc., and thus apply to the right half of the same cylinder, remembering that x is measured positive toward the left.

A temperature distribution such as has been assumed in this

section would be rarely met with. Its usefulness lies in the fact that it produces no distortion of the mid-plane of the cylinder. We may then superpose on this case the stresses caused by a temperature gradient in the axial direction only, obtaining any desired temperature distribution on the two surfaces of the shell, assuming always that it has circular symmetry. Fig. 14 shows the moments and loads applied to secure the condition of free ends for different cases of temperature distribution. These moments and loads produce the deflection y in this section.

TEMPERATURE GRADIENT IN AXIAL DIRECTION ONLY

Assume again a thin cylindrical shell of elastic material, of mean radius c and of length l . Let the temperature rise be a constant over any cross-section, but vary with the axial co-ordinate in a manner that may be expressed by the power series:

$$T = \frac{1}{2} \left[A + B \frac{x}{l} + C \frac{x^2}{l^2} + D \frac{x^3}{l^3} + \dots \right] \dots \dots [26]$$

Consider a unit strip cut from this shell as in Fig. 3. Let s_t be the average tensile stress in the tangential direction acting on the sides of the strip and y its displacement in the direction of the radius. The unit strain in the circumferential direction is:

$$e_t = \frac{y}{c} = \frac{s_t}{E} + \alpha T$$

Then

$$s_t = \frac{E y}{c} - E \alpha T$$

The force per unit length of each side of the strip is

$$s_t h = h \left(E \frac{y}{c} - E \alpha T \right)$$

The component of this force in the radial direction per unit length of strip is

$$\frac{s_t h}{c} = \frac{E h}{c} \alpha T - \frac{E h}{c^2} y$$

The change of sign indicates that for an outward deflection the force is inward. Referring to Fig. 5, there is now acting in addition to Ky the force $(Eh/c)\alpha T$ per unit length of strip, which is equivalent to a distributed load of that intensity acting on a beam supported by an elastic foundation. The differential equation becomes:

$$\frac{EI}{1 - \mu^2} \frac{d^4 y}{dx^4} + Ky = \frac{Eh}{c} \alpha T$$

The solution is, as before,

$$y = e^{-\beta x} (C_3 \cos \beta x + C_4 \sin \beta x) + \frac{ac}{2} \left[\left(A - \frac{6E}{\beta^4 l^4} - \dots \right) \right. \\ \left. + \left(B - \frac{30F}{\beta^4 l^4} - \dots \right) \frac{x}{l} + \left(C - \frac{90G}{\beta^4 l^4} - \dots \right) \frac{x^2}{l^2} + \dots \right]$$

Assume T to be given by the first four terms of the series in Equation [26]. From the condition that the ends are free, we find:

$$\frac{d^2 y}{dx^2} \Big|_{(x=0)} = 0 = -2\beta^2 C_4 + \frac{ac}{2} \frac{2C}{l^2}. \quad C_4 = \frac{ac}{2} \frac{C}{\beta^2 l^2}$$

$$\frac{d^3 y}{dx^3} \Big|_{(x=0)} = 0 = 2\beta^3 (C_3 + C_4) + \frac{ac}{2} \frac{6D}{l^3}. \quad C_3 = -\frac{ac}{2\beta^3 l^3} (\beta l C - 3D)$$

$$y = \frac{ac}{2} \left[A + B \frac{x}{l} + C \frac{x^2}{l^2} + D \frac{x^3}{l^3} + \frac{3D - \beta l C}{\beta^3 l^3} \theta + \frac{C}{\beta^2 l^2} \alpha \right]. [27]$$

$$\frac{d^2 y}{dx^2} = ac \left[\frac{C}{l^2} (1 - \phi) + \frac{3D}{\beta l^3} (\beta x + \alpha) \right]$$

$$\sigma_x = \frac{Eac}{1 - \mu^2} \eta \left[\frac{C}{l^2} (\phi - 1) - \frac{3D}{\beta l^3} (\beta x + \alpha) \right] \dots [28]$$

$$\sigma_t = \frac{Ea}{2} \left[\frac{3D - \beta l C}{\beta^3 l^3} \theta + \frac{C}{\beta^2 l^2} \alpha \right] + \mu \sigma_x \dots [29]$$

No stresses result from temperature gradients in this direction unless Equation [26] includes at least the second power of x . If no terms above the third power of x are present, the deflection curve away from the ends is the normal expansion curve of the cylinder wall for the temperature rise assumed.

As in the preceding section, these equations apply only to the left end of the cylinder. For the right end we must again use the constants A' , B' , etc., and measure x positive to the left.

By combining this case with the preceding one we may arrive at the desired distribution. There may also be superposed on these the effect of end connections or of pressure. Stresses due to gradients in the axial direction will be small, except in cases of localized heating as in welding.

INTERIOR TEMPERATURE A LINEAR FUNCTION OF x

Let us now consider cases in which a variable temperature gradient exists in the radial direction accompanied by a gradient along the mid-plane of the cylinder in the X -direction. These will be solved by superposing the solutions already obtained in these two cases. Assume the external temperature of the cylindrical shell to be kept at zero and the interior to be held at a value: $T_i = T_1 x/l$. In Equation [19], $B = T_1$; all other constants are zero. In Equation [26], the temperature along the midplane is: $T = (T_1/2)x/l$, so that $B = T_1$ in this equation also, and other constants are zero. Substitute these values of the constants in Equations [22] and [27], which may then be added to give:

$$y = \frac{1 + \mu}{2\beta^3} \frac{aT_1}{hl} \theta + \frac{acT_1}{2} \frac{x}{l}$$

In the same way, by adding Equations [23], [25], and [28],

$$\sigma_x = \frac{EaT_1}{1 - \mu} \frac{\eta}{h\beta l} (\beta x - \alpha)$$

And by adding [24], [25], and [29],

$$\sigma_t = \frac{EaT_1}{1 - \mu} \frac{\eta}{h\beta l} \left(\beta x - \mu \alpha + \frac{1 - \mu^2}{2\beta^2 c \eta} \theta \right)$$

These equations apply to the left end of the cylinder. For the right end we must substitute A' , B' , etc., to obtain:

$$y = \frac{1 + \mu}{2\beta^2} \frac{aT_1}{h} \left(\psi - \frac{\theta}{\beta l} \right) + \frac{aT_1}{2} \frac{c}{l} (l - x)$$

$$\sigma_x = \frac{EaT_1}{1 - \mu} \frac{\eta}{h} \left(1 - \phi - \frac{\beta x - \alpha}{\beta l} \right)$$

$$\sigma_t = \frac{EaT_1}{1 - \mu} \frac{\eta}{h} \left[1 - \frac{x}{l} - \mu \left(\phi - \frac{\alpha}{\beta l} \right) + \frac{1 - \mu^2}{2\beta^2 c \eta} \left(\psi - \frac{\theta}{\beta l} \right) \right]$$

When $\eta = h/2$ and $x = l/2$,

$$\sigma_x = \sigma_t = \frac{EaT_1}{4(1 - \mu)}$$

This value is necessarily the same as that obtained from the equations derived from the other end.

$$\text{When } x = 0, \sigma_x = 0, \text{ and } \sigma_t = \frac{EaT_1}{2} \left[1 + \frac{1 + \mu}{\beta^2 hc} \left(1 - \frac{1}{\beta l} \right) \right]$$

We should expect the values of the deflection and tangential stress at the end where $T_i = 0$ to be quite low, as may be shown by an example.

Example 4. In order to compare the result with Example 1 we shall again assume a cylinder of the same material and dimensions, but the internal temperature is to be given by:

$$T_i = T_1 \frac{x}{l}$$

where $T_1 = 100$ deg. Fahr., and the external temperature is zero.

The stresses at the middle are, of course, one-half those in

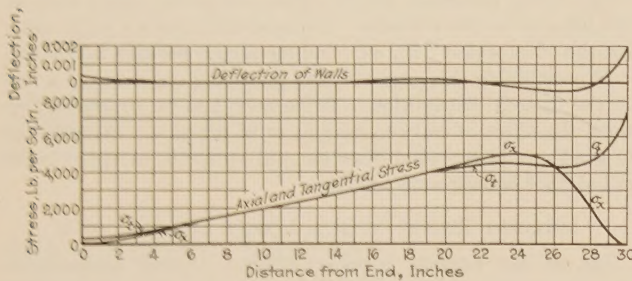


FIG. 15 CYLINDER OF EXAMPLE 4

the first example. The other results will differ at each end. At the end $T_i = 0$,

$$y = \frac{1 + \mu a T_1}{2 \beta^2 h l} = 0.182 \times 10^{-3} \text{ in.}$$

When $\eta = h/2$,

$$\sigma_t = \frac{EaT_1}{2} \frac{1 + \mu}{\beta^2 h c l} = 272 \text{ lb. per sq. in.}$$

At the end $T_i = T_1$,

$$y = \frac{1 + \mu a T_1}{2 \beta^2 h} \left(1 - \frac{1}{\beta l} \right) = 2.05 \times 10^{-3} \text{ in.}$$

When $\eta = h/2$,

$$\sigma_t = \frac{EaT_1}{2} \left[1 + \frac{1 + \mu}{\beta^2 hc} \left(1 - \frac{1}{\beta l} \right) \right] = 7575 \text{ lb. per sq. in.}$$

Fig. 15 shows graphically the stresses and deflection in this case.

INTERIOR TEMPERATURE A FUNCTION OF x^2

If in Equation [19] we place $C = T_1$ and all other constants equal to zero, we come to the case of parabolic distribution of the internal temperature in the direction of the axis. In Equation [26], if the temperature along the mid-plane is $T = T_1 x^2/l^2$, which makes $C = T_1$, and all other constants zero, we bring the exterior temperature to zero and the interior becomes $2T_1$.

After substituting these values for the constants we add Equations [22] and [27]:

$$y = \frac{aT_1}{1 - \mu} \frac{hc^2}{6l^2} + \frac{aT_1 c}{2\beta^2 l^2} (\beta^2 x^2 - \psi)$$

Adding [23], [25], and [28],

$$\sigma_x = \frac{EaT_1}{1 - \mu} \frac{\eta}{h l^2} \left(x^2 + \frac{hc}{1 - \mu} (1 - \phi) \right)$$

And from [24], [25], and [29],

$$\sigma_t = \frac{EaT_1}{1 - \mu} \frac{\eta}{h l^2} \left[x^2 + \frac{h^2 c}{6\eta} - \frac{h\psi}{2\beta^2 \eta} (1 - \mu) + \mu hc(\phi - 1) \right]$$

By substitution of the values of the constants applying to the right end of the cylinder we obtain the equations for stress and deflection in this part. In general, the curves of deflection and stresses will be similar to those in Fig. 15, except that the stress curve connecting the two end portions now becomes a parabola instead of a straight line.

CASES INVOLVING VARIATIONS OF PREVIOUS METHODS

Cases of sudden changes in wall thickness may be investigated in the same way as this problem is handled in the ordinary beam. The strip is cut at the point of change of section and a moment and shear assumed at each of the cut ends. The magnitude of these is then determined by the conditions of continuity at the junction. For a slightly tapering wall thickness an average value may be assumed, since the wall thickness plays only a small part in determining the stress.

A sudden change in the form of the function describing the interior temperature T_i might be handled in the same way.⁷ Usually such changes in the interior temperature will not occur suddenly, but merge into each other gradually, which requires no special treatment of the junction unless one portion be shorter than the half-wave-length, π/β .

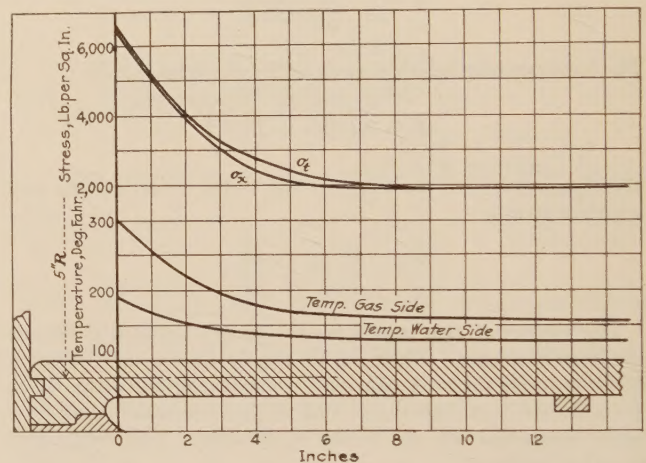


FIG. 16 TEMPERATURE AND STRESSES IN CYLINDER LINER—EXAMPLE 5

An example of this kind occurs in the cylinder liner of a Diesel engine. Near the combustion end the temperature of both the gas and water side are represented by equations of the second degree for distances of one-third to one-half the stroke. From this point on the curves become linear with a slight downward slope. Dr. Eichelberg (11) calculated these curves from a consideration of heat flow within the cylinder and later verified them by means of thermocouples embedded in the walls. Dr. Letson (12) also made experimental determinations of the temperature distribution in a liner, which agree with those of Dr. Eichelberg. Of course the maximum temperatures reached are far from being the same since these depend on the type of engine investigated, but the shape of the curves on both the gas and water sides were very similar.

Example 5. Fig. 16 shows a portion of a cylinder liner with temperature curves for the two surfaces as determined in the

⁷ Timoshenko and Lessells, "Applied Elasticity," p. 146.

experiments referred to. At the upper end the liner is pressed into its casing which is assumed to exert a moment and shear sufficient to bring the slope and deflection of the liner to zero at this point. At other points the liner is simply supported, no moment being exerted. The material is cast iron and the dimensions are: mean diameter, 10 in.; thickness, 1 in.; complete length, possibly 25 to 30 in. The temperature of the liner on the gas side is given by

$$T_g = 300 - 50x + 5x^2$$

and on the water side by

$$T_w = 190 - 20x + 1.8x^2$$

The moment equation becomes, for the region between $0 < x < 7$:

$$\frac{EI}{1-\mu^2} \frac{d^2y}{dx^2} + \int_0^x Ky(x-\mu)d\mu + M_0 + q_0x = \frac{EI}{1-\mu} \frac{a}{h} (T_g - T_w)$$

The solution of this equation yields the following results:

$$M_0 = 1004 \text{ lb. in. per in. of circumference.}$$

$$q_0 = -294 \text{ lb. in. per in. of circumference.}$$

$$y = 0.1066(1-\mu)10^{-3} \text{ in.}$$

For $\eta = h/2$,

$$\sigma_x = 60(110 - 30x + 3.2x^2) - 191.2\psi$$

$$\sigma_t = 60(110 - 30x + 3.2x^2) + 319.8(1-\phi) - 47.8\psi$$

The temperature of the mid-plane in this case will be, $T = 55 - 15x + 1.6x^2$. Stresses resulting from the distortion of this plane will be

$$\sigma_x = 2 \frac{Eac}{1-\mu^2} \eta \left[\frac{1.6}{l^2} (\phi - 1) \right]$$

$$\sigma_t = Ea \frac{1.6}{\beta^2 \eta^2} \alpha + \mu \sigma_x$$

The maximum value of σ_x in the above equations is -75 lb. per sq. in. in this example, and for σ_t is 36 lb. per sq. in. The effect of the distortion of the mid-plane on stresses in this case is therefore small.

Beyond the point $x = 7$ the temperature curve becomes a straight line, and its downward slope is so small that it may be neglected. The stress in this portion is then given by $Ea(T_g - T_w)/2(1-\mu)$, which corresponds with the value of σ_x taken from the equation above for the point $x = 6$. In the stress curves of Fig. 16 it is noted that σ_t is slightly higher for this value of x , which is owing to the effect of the constant deflection produced by the parabolic temperature distribution.

Because of the effect of the moment and shear at the end in limiting the deflection, the maximum stress in this case is only 150 lb. per sq. in. above that calculated by the usual formulas applying to long cylinders with temperature varying only in the radial direction. Example 2 indicated similar results. The general effect of such moments and shears is, of course, to approximate the conditions of a long cylinder, but whatever the value of external forces or moments applied, the method developed in this paper enables the resulting stresses to be easily calculated.

Appendix No. 1

FROM the curves of Figs. 17 and 18 may be taken the values of the expressions represented by ϕ , ψ , θ , and α , which are required for the calculation of stresses and deflection of the

cylinder wall. The derivatives of these functions are required in many calculations and are given below.

$$\frac{d\theta}{dx} = -\beta\phi; \quad \frac{d\alpha}{dx} = \beta\psi; \quad \frac{d\phi}{dx} = -2\beta\alpha; \quad \frac{d\psi}{dx} = -2\beta\theta$$

$$\frac{d^2\theta}{dx^2} = 2\beta^2\alpha; \quad \frac{d^2\alpha}{dx^2} = -2\beta^2\theta; \quad \frac{d^2\phi}{dx^2} = -2\beta^2\psi; \quad \frac{d^2\psi}{dx^2} = 2\beta^2\phi$$

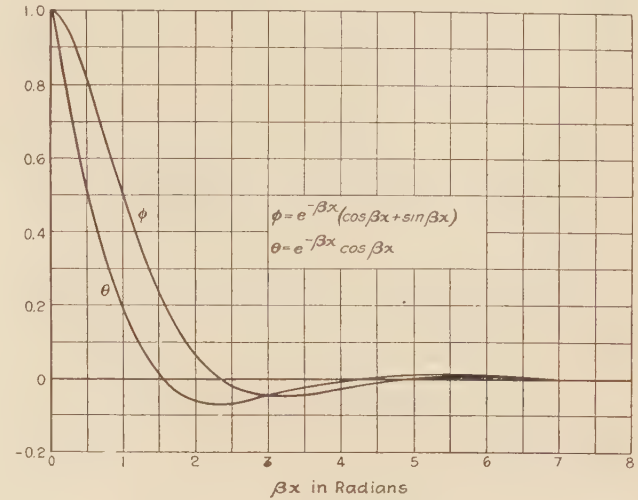


FIG. 17 VALUES OF THE FUNCTIONS ϕ AND θ

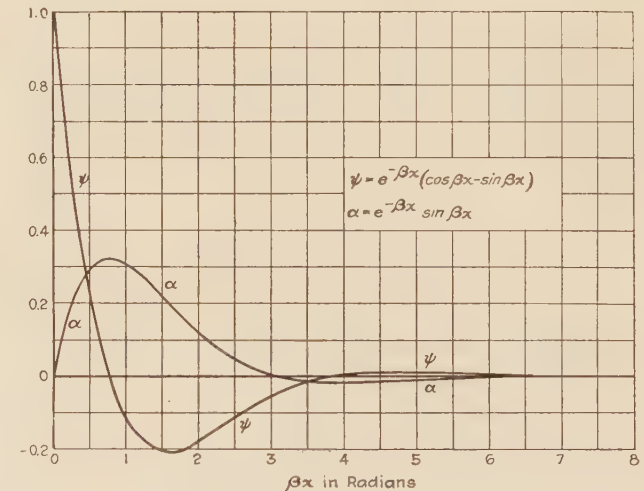


FIG. 18 VALUES OF ψ AND α

Appendix No. 2

ALTERNATE DERIVATION OF FORMULA [9]

Fig. 2(a) shows a portion of a thin cylinder which is assumed to be infinitely long. Its internal surface is held at the constant temperature T_1 while the outside is kept at zero. The resulting axial and tangential stresses are the same as those in a thin plate, similarly heated and not allowed to bend, that is, $\frac{EaT_1}{1-\mu} \frac{\eta}{h}$.

We note that at the ends of the portion assumed, the axial stresses are such as to produce a moment whose value is $\frac{EI}{1-\mu} \frac{aT_1}{h}$.

If we wish to investigate a cylinder of finite length we can produce a condition of free ends by imagining a moment of opposite direction applied to them as in Fig. 2(b). The stresses pro-

duced by this moment may then be added to those already existing.

To investigate the stresses produced by a moment acting on the ends of such a cylinder, we assume an infinitesimal parallelo-piped cut from its wall as in Fig. 19.* In this figure the expression $\sigma_r + \dots$ is an abbreviation for $\sigma_r + \frac{\partial \sigma_r}{\partial r} dr$, and similarly for the stresses σ_x and τ .

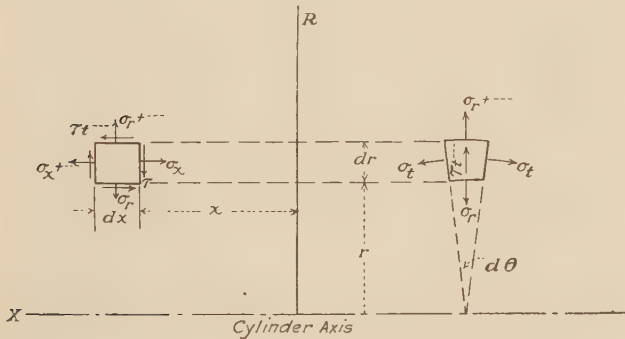


FIG. 19

The conditions of equilibrium in the radial and axial directions give the two equations:

$$r \frac{\partial \sigma_x}{\partial x} + r \frac{\partial \tau}{\partial r} + \tau = 0 \dots \dots \dots [30]$$

$$\sigma_r + r \frac{\partial \sigma_r}{\partial r} + r \frac{\partial \tau}{\partial x} - \sigma_t = 0$$

Let each of these be multiplied by dr and integrated between the limits a and b , which are the internal and external radii of the cylinder. We note that σ_r and τ are zero at each of these limits. Then:

$$\int_a^b \frac{\partial \sigma_x}{\partial x} r dr = 0 \dots \dots \dots [31]$$

$$\int_a^b \frac{\partial \tau}{\partial x} r dr - \int_a^b \sigma_t dr = 0 \dots \dots \dots [32]$$

Now multiply [30] by rdr and integrate in the same way:

$$\int_a^b \frac{\partial \sigma_x}{\partial x} r^2 dr - \int_a^b \tau r dr = 0$$

Take the partial of this with respect to x :

$$\int_a^b \frac{\partial^2 \sigma_x}{\partial x^2} r^2 dr - \int_a^b \frac{\partial \tau}{\partial x} r dr = 0$$

and from [32] we have

$$\int_a^b \frac{\partial^2 \sigma_x}{\partial x^2} r^2 dr = \int_a^b \sigma_t dr \dots \dots \dots [33]$$

Now consider a strip cut from this cylinder as in Fig. 3, whose width is $cd\theta$. The condition that the axial stress over any cross-section of the strip shall form a couple, is:

$$\int_a^b \sigma_x r d\theta dr = 0$$

The agreement of this equation with [31] indicates that this is a possible stress condition. The moment of this couple about any point is:

$$M = - \int_a^b \sigma_x r^2 d\theta dr \dots \dots \dots [33]$$

by choosing a point on the cylinder axis. Then from Equation [33]:

$$\frac{d^2 M}{dx^2} = - d\theta \int_a^b \sigma_t dr \dots \dots \dots [34]$$

Now by geometry $e_t = y/c$, and from the equations expressing Hooke's law for the two-dimensional stress condition,

$$e_t = \frac{\sigma_t}{E'} - \mu \frac{\sigma_x}{E}$$

where e_t is the unit tangential deformation. Then⁸

$$\sigma_t = E' \frac{y}{c} + \mu \sigma_x$$

and since

$$\int_a^b \sigma_x dr = 0, \quad d\theta \int_a^b \sigma_t dr = E' \frac{y}{c} d\theta \int_a^b dr = E' \frac{y}{c} h d\theta$$

Equation [34] becomes:

$$\frac{d^2 M}{dx^2} = - E' \frac{y}{c} h d\theta \dots \dots \dots [35]$$

For the bending of a thin strip, we have⁸

$$M = \frac{EI}{1 - \mu^2} \frac{d^2 y}{dx^2}$$

where $I = cd\theta h^3/12$. Then from [35]:

$$\frac{d^4 y}{dx^4} + \frac{12(1 - \mu^2)}{h^2 c^2} y = 0 \dots \dots \dots [9]$$

and we have as before:

$$y = e^{-\beta x} (C_3 \cos \beta x + C_4 \sin \beta x)$$

At the ends the shear is zero and the moment is $M_0 = \frac{EI}{1 - \mu} \frac{dT_1}{h}$

By evaluating $\frac{d^2 y}{dx^2(x=0)}$ and $\frac{d^3 y}{dx^3(x=0)}$, we find $C_3 = - C_4 = \frac{aT_1}{h} \frac{1 + \mu}{2\beta^2}$.

The axial and tangential stresses produced by the application of the moment alone are then found to be

$$\sigma_x' = - \frac{EaT_1}{1 - \mu} \frac{\eta}{h}$$

$$\sigma_t' = \frac{EaT_1}{1 - \mu} \frac{\eta}{h} \left(\frac{1 - \mu^2}{2\beta^2 c \eta} - \mu \right)$$

The total stress in the cylinder walls will be found by adding to these the stress already existing owing to the temperature gradient. This, of course, reduces σ_x to zero at the ends and gives for σ_t , as before,⁸

$$\sigma_t = \frac{EaT_1}{h} \left(\frac{1 + \mu}{2\beta^2 c} + \eta \right)$$

BIBLIOGRAPHY OF THERMAL STRESSES

PERIODICALS

1. T. M. C. Duhamel, "Memorie sur le Calcul des Actions Moleculaires etc.," Académie des Sciences de L'institute de France, vol. 5 (1838), p. 440.

⁸ Contrast with Föppl, "Vorlesungen über Technische Mechanik," vol. 5, p. 248.

2. Franz Neumann, "Die Gesetze der Doppelbrechung des Lichts," Abhandlungen Akademie der Wissenschaften zu Berlin, 1841, pp. 3-247.
3. C. W. Borchardt, "Untersuchungen über die Elasticität fester isotroper Körper unter Berücksichtigung die Wärme," Monatsberichte, Akademie der Wissenschaften zu Berlin, 1873, pp. 9-56.
4. J. Hopkinson, "On the Stresses Caused in an Elastic Solid by Inequalities of Temperature," *The Messenger of Mathematics*, vol. 8 (1879), p. 168.
5. Alfons Leon, "Spannungen und Formänderungen eines Holzyinders und einer Hohlkugel, etc.," *Zeitschrift für Mathematik und Physik*, vol. 52 (1905), pp. 174-190.
6. Rudolph Lorenz, "Temperaturspannung in Holzyinder," *Zeit. V.D.I.*, vol. 51 (1907), pp. 743-747.
7. Adams and Williamson, "Temperature Distribution in Solids During Heating or Cooling," *Physical Review*, vol. 14 (1919), pp. 99-114.
8. Williamson, "Strains Due to Temperature Gradients, etc.," *Journal Washington Academy of Sciences*, vol. 9 (1919), pp. 209-217.
9. Adams and Williamson, "The Annealing of Glass," *Journal Franklin Institute*, vol. 190 (1920), pp. 597-632 and 835-870.
10. C. H. Lees, "The Thermal Stresses in Solid and Hollow Circular Cylinders Concentrically Heated," *Proc. Royal Soc. of London*, vol. 101 (1922), p. 411.
11. G. Eichelberg, "Temperaturverlauf und Wärmespannungen in Verbrennungsmotoren," *Forschungsarbeiten V.D.I.*, Heft 263 (1923).
12. H. F. G. Letson, "Temperature Distribution in Diesel-Engine Liners," *Proc. Inst. M. E.*, London (1925), pp. 19-52.
13. G. Grünberg, "Über—durch ungleichförmige Erwärmung erregten Spannungszustand," *Zeitschrift für Physik*, vol. 35 (1925), pp. 548-555.
14. W. Nusselt, "Der Wärmeübergang in den Diesel-maschine," *Forschungsarbeiten V.D.I.*, Heft 14, Band 70 (1926), p. 468.
15. R. Sulzer, "Temperature Variations and Heat Stresses in Diesel Engines," *Engineering*, vol. 121 (1926), p. 447.
16. L. H. Barker, "Stresses in Tubes," *Engineering*, vol. 124 (1927), p. 443.
17. K. Requa, "Beitrag zur Beurteilung von Temperaturefeld, etc.," *Forschungsarbeiten V.D.I.*, Heft 301 (1928).
18. A. Nägel, "Transfer of Heat in Reciprocating Engines," *Engineering*, vol. 127 (1929), pp. 59, 179, 279, 466, 626.

BOOKS

20. Franz Neumann, "Vorlesungen über die Theorie der Elasticität," (1885), pp. 107-120.
21. Föppl, A., "Vorlesungen über Technische Mechanik," vol. V, 4th Ed., p. 238.
22. Case, "Strength of Materials," (1925), p. 456.
23. Timoshenko and Lessells, "Applied Elasticity," (1925), pp. 102, 146, 272.
24. Love, "Theory of Elasticity," 4th Ed., (1927), p. 108.
25. Stodola, "Steam and Gas Turbines," (1927), pp. 388, 1155.
26. Timoshenko, "Strength of Materials," (1930), pp. 487, 550.
27. Carslaw, "The Conduction of Heat," (1921).
28. Ingersoll and Zobel, "The Mathematical Theory of Heat Conduction," (1913).
29. Byerly, "Fourier Series and Spherical Harmonics," (1893).

